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ON p-NILPOTENT GROUPS WITH EXTREMAL p-BLOCKS

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Let G be a finite group, and p a fixed prime number. It is well known that

(I) if G is p-closed, then every p-block of G has full defect, and

(II) if G has the pTI-property, then every p-block of G has either full defect or defect zero.

Here, " G is p-closed " means that a Sylow p-subgroup of G is normal, and " G has the pTI-property " means that the intersection of two distinct Sylow p-subgroups of G is the identity. It is interesting to consider each converse of (I) and (II). In general, neither the converse of (I) nor of (II) is true. In fact, the Mathieu groups M_{22} and M_{24} have only one 2-block, but these groups are neither 2-closed nor have the 2TI-property. Moreover, if H is a p-solvable group of p-length greater than 1, then $G = H/O_{p'}(H)$ has only one p-block, but G is neither p-closed nor has the pTI-property. In case $p = 2$, several authors studied this problem. In what follows, we call G a pFD-group if every p-block of G has full defect. On the other hand, we call G a pFZD-group if every p-block of G has either full defect or defect zero.

In [2], Harada proved the following

Theorem 1 ([2, Theorem 1]). Suppose that a Sylow 2-subgroup of G is abelian. Then G is 2-closed if and only if it is a 2FD-group.

In [3], Herzog has generalized the above theorem as follows:

Theorem 2 ([3, Theorems 1 and 2]). (1) A group G is 2-closed if and only if it is a 2FD-group and every intersection of two distinct Sylow 2-subgroups of G is centralized by a Sylow 2-subgroup of G.

(2) A group G has the 2TI-property if and only if it is a 2FZD-group and every intersection of two distinct Sylow 2-subgroups of G is centralized by a Sylow 2-subgroup of G.

Furthermore, in [1], Chillag and Herzog proved the following theorems.

Theorem 3 ([1, Theorems 1 and 2]). Suppose that G is a 2-nilpotent group and a Sylow 2-subgroup of G is a dihedral group, a generalized quaternion group or a quasidihedral group. Then there holds the following:

(1) G is 2-closed if and only if it is a 2FD-group.

(2) G has the 2TI-property if and only if it is a 2FZD-group.

Theorem 4 ([1, Theorem 4]). Suppose that a Sylow 2-subgroup of G is a quaternion group of order 8. Then there holds the following:

(1) G is 2-closed if and only if it is a 2FD-group.

(2) G has the 2TI-property if and only if it is a 2FZD-group.

Here, we report that each converse of (I) and (II) is true if G is a p-nilpotent group. At first, the following proposition is an immediate consequence of [5, Theorem 4].

Proposition. Let G be a p-nilpotent group with a normal p-complement N. Then G is a pFD-group if and only if, for every $x \in N$, $C_G(x)$ contains a Sylow p-subgroup of G.

By making use of the proposition, we can easily obtain the following

Theorem 5. Let G be a p-nilpotent group. Then G is p-closed if and only if it is a pFD-group.

Let H be a normal subgroup of G such that $|G/H|$ is relatively prime to p. Then by [4, Proposition 4.2], we see that if G is a pFD-group then H is also a pFD-group. Hence, we get the following, which contains [2, Lemma 1].

Corollary 1. Let G be a p-solvable group. Then G is p-closed if and only if it is a pFD-group and has p-length 1.

Furthermore, by making use of Theorem 5, we can prove the following

Theorem 6*). Let G be a p-nilpotent group, and $P \cap Q$ an intersection of maximal order of two distinct Sylow p-subgroups of G. Then there exists a p-block of G with defect group $P \cap Q$.

As a corollary to this theorem, we get the following

Theorem 7. Let G be a p-nilpotent group. Then G has the pTI-property if and only if it is a pFZD-group.

Let G be a p-solvable group. If G has the pTI-property, then $G/O_{p'}(G)$ also has the pTI-property, and hence G has p-length 1. Therefore, Theorem 7 together with [4, Proposition 4.2] implies the following

Corollary 2. Let G be a p-solvable group. Then G has the pTI-property if and only if it is a pFZD-group and has p-length 1.

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*) This theorem was suggested by Dr. T. Okuyama.

References

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